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MOURRE THEORY FOR TIME-PERIODIC SYSTEMS

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We consider the following Schrödinger equation with time-dependent Hamiltonian on $\mathbb{R}^{\nu}$,

$$i\frac{\partial}{\partial t}u(t,x)=H(t)u(t,x), \qquad (t,x)\in\mathbb{R}\times\mathbb{R}^{\nu}, \tag{1}$$

$$H(t)=-\Delta_x+V(t), \tag{2}$$

where $V(t)$ is a multiplicative operator by a function $V(t,x)$ which is periodic in t with period 2π:

$$V(t+2\pi,x)=V(t,x). \tag{3}$$

As is well-known, with some suitable conditions on $V(t,x)$, $H(t)$ generates a unique unitary propagator $\{U_1(t,s)\}_{-\infty<t,s<\infty}$. For $H_0=-\Delta_x$, the associated unitary propagator is denoted by $U_0(t,s)=e^{-i(t-s)H_0}$. A traditional way to study the temporal asymptotics as $t\to\pm\infty$ of $U_1(t,s)$ is to introduce an operator $K=-i\frac{d}{dt}+H(t)$ on $\mathbb{T}\times\mathbb{R}^{\nu}$, where $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$, and to investigate the asymptotic behavior of $e^{-i\sigma K}$. They are mutually related through the following formula

$$(e^{-i\sigma K}f)(t,x)=(U_1(t,t-\sigma)f(t-\sigma,\cdot))(x), \tag{4}$$

for $f\in\mathbb{H}=L^2(\mathbb{T}\times\mathbb{R}^{\nu})$. Let

$$K_0=-i\frac{d}{dt}+H_0. \tag{5}$$

Definition 1 (conjugate operator).

$$A=\frac{1}{2}(L_D\cdot x+x\cdot L_D) \tag{6}$$

where $D_x=\frac{1}{i}\nabla_x$ *and* $L_D=(L_j)_{1\le j\le\nu}$ *with* $L_j=D_{x_j}<D_x>^{-2}$.

The following assumption is imposed on $V(t)$.

Assumption 1. *Let V be the operator of multiplication by the function $V(t,x)$ on $\mathbb{H}$. We assume that*

(i) V, $[V,A]$ *are extended to K_0-compact operators.*

(ii) $[[V,A],A]$ *is extended to a K_0-bounded operator.*

We denote the extension of the form $[K, A]$ as $[K, A]^0$.

Theorem 1. *Suppose Assumption 1 is satisfied. For $\lambda \in \mathbb{R} \setminus \mathbb{Z}$, let $d(\lambda, \mathbb{Z})$ denote the distance from λ to $\mathbb{Z}$. Then, Eigenvalues of K(the set of which are denoted by $\sigma_{pp}(K)$) are discrete with possible accumulation points in $\mathbb{Z}$. If $\lambda \in \mathbb{R} \setminus (\mathbb{Z} \cup \sigma_{pp}(K))$, for each $\epsilon > 0$ there exists $0 < \delta < d(\lambda, Z)$ such that*

$$f(K)i[K, A]^0 f(K) \geq (\frac{2d(I, \mathbb{Z})}{d(I, \mathbb{Z}) + 1} - \epsilon) f(K)^2 \tag{7}$$

for all $f \in C_0^\infty([\lambda - \delta, \lambda + \delta])$.

Let $\mathfrak{B}(\mathbb{H})$ be the set of bounded operators on $\mathbb{H}$.

Theorem 2. *Suppose $\alpha > 1/2$.*

(i) *For each closed interval $I \subset \mathbb{R} \setminus (\mathbb{Z} \cup \sigma_{pp}(K))$ the following inequalities hold:*

$$\sup_{Imz \neq 0, Rez \in I} \| < x >^{-\alpha} (K - z)^{-1} < x >^{-\alpha} \|_{\mathfrak{B}(\mathbb{H})} < \infty. \tag{8}$$

(ii) *There exist the norm limits in $\mathfrak{B}(\mathbb{H})$.*

$$\lim_{Imz \to 0, Rez \in I} < x >^{-\alpha} (K - z)^{-1} < x >^{-\alpha} .$$

$< x >^{-\alpha} (K - \lambda \mp i0)^{-1} < x >^{-\alpha}$
is Hölder continuous with respect to $\lambda \in \mathbb{R} \setminus (\mathbb{Z} \cup \sigma_{pp}(K))$.

Next we proceed to the propagation estimates. We need the following stronger assumption on the potential.

Assumption 2. *There exists $\delta_0 > 0$ such that*

$$V(t, \cdot) \in C(\mathbb{T}; C^\infty(\mathbb{R}^\nu)), \quad |\partial_x^\alpha V(t, x)| \leq C_\alpha < x >^{-\delta_0 - |\alpha|}, \quad {}^\forall \alpha. \tag{9}$$

Theorem 3. *Suppose Assumption 2 is satisfied. Let $E \in \mathbb{R} \setminus (\mathbb{Z} \cup \sigma_{pp}(K))$, and $\epsilon > 0$ be given. Then there exists a small open interval I containing E such that for any $f \in C_0^\infty(I)$ and $s' > s > 0$,*

(10)
$$\|\chi\Big(\frac{|x|^2}{4\sigma^2} - \frac{d(I, \mathbb{Z})}{d(I, \mathbb{Z}) + 1} < -\epsilon\Big) e^{-i\sigma K} f(K) < x >^{-s'} \|_{\mathfrak{B}(\mathbb{H})} = O(\sigma^{-s}) \quad as \quad \sigma \to \infty$$

where $\chi(x < a)$ denotes the characteristic function of the interval $(-\infty, a)$.